ORC 67-9
February 1967

OPTIMAL ASSIGNMENTS IN AN ORDERED SET; AN APPLICATION OF MATROID THEORY

by

David Gale

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF CALIFORNIA - BERKELEY

OPTIMAL ASSIGNMENTS IN AN ORDERED SET; AN APPLICATION
OF MATROID THEORY

by

David Gale
Operations Research Center
University of California, Berkeley

February 1967 ORC 67-9

This research was supported by the Office of Naval Research under Contract Nonr-222(83), and with the University of California, Berkeley. Reproduction in whole or part is permitted for any purpose of the United States Government.

OPTIMAL ASSIGNMENTS IN AN ORDERED SET; AN APPLICATION OF MATROID THEORY

by

David Gale

1. PREAMBLE

The substance of this paper is an observation concerning a certain rather natural type of assignment problem which is described in the next section. After completing an earlier version of the exposition of this result, I was introduced by D. R. Fulkerson to the concept of matroid and realized that what I had observed was a rather obvious property of these objects, so that for those familiar with the matroid literature my theorem could be proved in a few lines. On the other hand, since the result itself may be of interest to people who have not been initiated into matroid lore I decided to prepare the present revised version which may also serve as an introduction to matroids by means of this particular application.

2. THE ASSIGNMENT PROBLEM

A certain set of jobs has been arranged in order of importance by some priority system and it is desired to fill the jobs from a pool of workers where each worker is qualified for some subset of the jobs. In general, it will not be possible to fill all of the jobs and the problem is therefore to choose the set of jobs to be filled in some optimal way. Roughly speaking, given all possible assignable sets of jobs one wishes to choose the one with "highest priority." It is not clear, however, what this means as, for example, if the choice were between filling jobs 1, 4, 6 or 2, 3, 4, 5. The purpose of this note is to show that, in fact, there always does exist a "best" assignment, meaning an assignment of jobs $J_1, \ldots, J_n$ such that any other possible assignment can be arranged in some order $J_1', \ldots, J_m'$ such that J_i has at least as high a priority as J_i'.

We formalize the problem as follows: Let X and Y be finite sets and let Φ be a function from X to subsets of Y. In our example X represents the jobs, Y the workers and $y \in \Phi(x)$ means y is qualified for x.

A subset $A \subset X$ is <u>assignable</u> if there is a univalent function ϕ, called an <u>assignment</u>, from A to Y such that $\phi(x) \in \Phi(x)$ for all x in A. Let $\mathcal{a}$ be the family of all assignable subsets of X.

Now suppose X to be totally ordered and let $\mathfrak{F}$ be any family of subsets of X. We call a set A of $\mathfrak{F}$ <u>optimal</u> if for any other set B of $\mathfrak{F}$ there is a univalent mapping f from B to A such that $f(b) \geq b$ for all b in B. Our result is, then,

THEOREM 1. The family $\mathcal{a}$ of assignable sets of X contains an optimal element.

3. MATROIDS AND ASSIGNABLE SETS

DEFINITION. A family $\mathfrak{F}$ of subsets of a finite set X is a <u>matroid</u> provided the following conditions hold:

(1) If $A \in \mathfrak{F}$ and $B \subset A$ then $B \in \mathfrak{F}$.

(2) For any $X' \subset X$ all elements of $\mathfrak{F}$ which are maximal in X' have the same cardinality.

THEOREM 2. The family $\mathcal{a}$ of assignable sets is a matroid.

This theorem is a consequence of a more general result on graphs due to Edmonds and Fulkerson [1]. For the sake of completeness, we present our original proof for this special case.

<u>Proof</u>. Condition (1) is obvious. To prove (2) we denote by $|S|$ the cardinality

of the set S and give a proof by induction on $|X'|$, the conclusion for $|X'| = 1$ being obvious. Suppose now that A and A' are both subsets of X' and elements of $\mathcal{a}$ and that $|A| < |A'|$. We must then find B in $\mathcal{a}$ which is contained in X' and properly contains A. Let ϕ be an assignment of A and ϕ' an assignment of A'.

<u>Case I</u>. There exists a' in $A' - A$ such that $\phi'(a') \notin \phi(A)$. Then let $B = A \cup a'$ and extend ϕ to B by defining $\phi(a') = \phi'(a')$.

<u>Case II</u>. $\phi'(A' - A) \subset \phi(A)$. Then there exists a' in $A' \cap A$ such that $\phi'(a') \notin \phi(A)$ since $|A'| > |A|$. Now ϕ and ϕ' are assignments from the set $X - a'$ to $Y - \phi'(a)$ and by inductive hypothesis therefore there is a set B' of $\mathcal{a}$ in $X' - a'$ which properly contains $A - a'$ since $|A - a'| < |A' - a'|$. Let ϕ'' be an assignment of B' and extend it to $B = B' \cup a'$ by defining $\phi''(a') = \phi'(a')$. This completes the proof.

4. MATROIDS ON ORDERED SETS

The proof of Theorem 1 is now a direct consequence of Theorem 2 and

THEOREM 3. If $\mathfrak{F}$ is a matroid on an ordered set then $\mathfrak{F}$ contains an optimal element.

<u>Proof</u>. Consider all sets of maximal cardinality in $\mathfrak{F}$ and let A be the set which is <u>lexicographically maximum</u> among them. This means that if

$$A = [a_1, \ldots, a_n]$$

$$B = [b_1, \ldots, b_n]$$

are distinct maximal sets with elements listed in decreasing order and then $a_i > b_i$ where i is the smallest index such that $a_i \neq b_i$.

We claim A is optimal, for let B be any other set

$$B = [b_1, \ldots, b_m]$$

with elements listed in decreasing order. If $b_i \leq a_i$ then A is optimal. If not then, say, $b_r > a_r$. But then consider the set $S = [a_1, \ldots, a_{r-1}, b_1, \ldots, b_r]$. Now S contains subsets $[a_1, \ldots, a_{r-1}]$ and $[b_1, \ldots, b_r]$ both in $\mathfrak{F}$, hence by (2) the set $[a_1, \ldots, a_{r-1}, b_i]$ is in $\mathfrak{F}$ for some i . Again by (2) this set lies in an n element set A' of $\mathfrak{F}$, but A' is then lexicographically greater than A , a contradiction.

Finally, we observe a converse to Theorem 3 which gives a characterization of matroids.

THEOREM 4. Let $\mathfrak{F}$ be a family of subsets of a finite set X which satisfies (1) and

(3) for any ordering of X, $\mathfrak{F}$ contains an optimal set. Then $\mathfrak{F}$ is a matroid.

Proof. If $\mathfrak{F}$ is not a matroid then there exist sets A and A' in $\mathfrak{F}$ where $|A| < |A'|$ and A is maximal in $A \cup A'$. Then order the elements of X so that the first r elements $a_1, \ldots, a_r$ make up $A \cap A'$, the next s elements $b_1, \ldots, b_s$ make up $A - A'$ and the next t elements $c_1, \ldots, c_t$ make up $A' - A$, where we note that $t > s$. The remaining elements we denote by $d_1, \ldots, d_k$. Now clearly if $\mathfrak{F}$ has an optimal set it must be the lexicographical maximum and clearly this maximum listed in decreasing order is the set

$$\tilde{A} = [a_1, \ldots, a_r, b_1, \ldots, b_s \text{ and possibly some of the } d_i] .$$

But A cannot be optimal because it does not "dominate" $A' = [a_1, \ldots, a_r, c_1, \ldots, c_t]$, i.e., there is no univalent mapping f from A' to $\tilde{A}$ such that $f(a') \geqq a'$, for either $\tilde{A} = A$ and there is no univalent mapping at all or $f(a') = d_i$ for some a' in A', in which case $f(a') < a'$.

Remarks. A second application of Theorem 3 is to the problem of finding the minimum spanning tree in a graph. P. Rosenstiehl has observed [2] that this well-known problem also has a solution assuming only that the edges of the graph are ordered. (The usual ordering of edges is, of course, the one given by their lengths.) Since the trees in a graph are well known to form a matroid on the set of edges, Rosenstiehl's result is a special case of Theorem 3.

Finally, we remark on the problem of efficient computational methods for finding optimal sets. From Theorem 3 it is seen that this is the problem of finding the lexicographic maximum. We can thus assemble the optimum set in a pointwise fashion. The largest element of X which belongs to $\mathfrak{F}$ is first chosen. Having chosen k elements $a_1, \ldots, a_k$, we then choose the largest element a_{k+1} such that $[a_1, \ldots, a_{k+1}]$ is in $\mathfrak{F}$. The problem is then one of being able to decide when a given set belongs to $\mathfrak{F}$. For the case of trees, the answer is simply that the set contain no cylces; and we are lead at once to the first algorithm of Krushal [3]. For the case of assignments, one has the famous criterion of Hall [4] which has been incorporated into an efficient algorithm by Kuhn [5].

BIBLIOGRAPHY

[1] Edmonds, J., and D. R. Fulkerson, "Transversals and Matroid Partition," Journ. of Res., Nat. Bureau of Standards, Vol. 69B, No. 3, pp. 147-153, (July-September 1965).

[2] Rosenstiehl, P., "L'arbre Minimum d'un Graph," forthcoming, Actes des Journees sur la Theorie des Graphs et ses Applications, Dunod, Paris, (1967).

[3] Kruskal, J. B., "On the Shortest Spanning Subtree of a Graph," Proc. Amer. Math. Soc., Vol. 7, p. 48, (1956).

[4] Hall, P., "On Representations of Subsets," Journ. London. Math. Soc., Vol. 10, pp. 26-30, (1935).

[5] Kuhn. H.W., "The Hungarian Method for Solving the Assignment Problem," Naval Research Logist. Quart., Vol. 2, pp. 83-97, (1955).